Robust Controller Design Using Frequency Domain Constraints

R.D. Hefner*
The Aerospace Corporation, El Segundo, California
and
D.L. Mingori†
University of California, Los Angeles, California

This paper describes a method for designing a controller with improved robustness with respect to truncated flexible modes. The approach involves minimization of a quadratic performance index subject to constraints in the frequency domain. The frequency domain criteria are chosen so as to sufficiently attenuate the high frequency response of the full dynamic system while attempting to maintain the overall performance of the closed-loop system. The resulting constraint relationships are cast into a functional minimization framework and parameter optimization techniques are used to determine the solution.

Introduction

THE application of optimal control theory to physical problems relies on the fidelity of the mathematical model used to describe the physical system. Since a perfect model can never be obtained for any physical process, the sensitivity of the control design to modeling errors and parameter variations is always a key issue. In general, these modeling errors can be broken into four types: errors in model order, errors due to neglected disturbances, errors due to neglected nonlinearities, and parameter errors.[1]

For large, flexible space structures, one of the most prominent sources of modeling error is the deletion of modes in the formation of the design model. The motivation for this modal truncation can be from either modeling or control design considerations. Modeling problems arise from the fact that the dynamic characteristics (natural frequencies and mode shapes) of complicated space structures cannot be determined with the same accuracy as for simple structures, such as uniform beams or membranes. A spacecraft model is usually constructed with a structural analysis problem such as NASTRAN and relies on the finite element method[2-4] or spatial discretization.[5,6] This approach, referred to as substructure synthesis,[7] requires that the overall structure be broken into separate substructures which are modeled individually and then reassembled into a total system model. Spatial discretization of each substructure separately implies a truncation at that level, with each infinite-dimensional subsystem being replaced by a finite-dimensional one.

Control design problems result from the computational difficulties inherent in the application of control algorithms to high-order models. Although a wide variety of techniques exist for the control of dynamic systems, their practical uses have primarily been restricted to low-order models. In general, these approaches are subject to greatly increasing complexity and computer run time with decreasing computational accuracy as the model size increases. Because of this, modal truncation often occurs at a system level in order to obtain a reasonably sized design model. This process is often guided by modeling limitations, i.e., the low-frequency (more accurately known) modes are retained, and the high-frequency modes are deleted.

Because of the light damping present in space structures, and their severe performance requirements, these modeling errors can cause significant problems. A control system design based on a reduced-order model must be used with an infinite-dimensional physical system. In the full-order, closed-loop system, the unmodeled, residual mode effects, commonly called control and observation spillover, can cause a severe loss of performance or even instability.[8,9]

Several authors have studied techniques for synthesizing controllers that are robust with respect to these types of modeling errors, either through better model truncation techniques or better controller design procedures. This paper proposes a controller design procedure that meets classical frequency domain constraints within the framework of a modern, optimal control design. This permits sufficient attenuation of the high-frequency response of the full dynamic system while maintaining overall closed-loop performance. Although the numerical example given will illustrate the technique for the case of truncated modes, the procedure presented is general enough to handle other classes of modeling error.

Robust Controller Design

The problem to be considered is the design of a control system for a highly flexible space structure. It is assumed that a reduced-order design model of this structure has been developed through the techniques discussed earlier, and is of the form

$$\dot{x}_D = A_D x_D + B_D u + D_D v \tag{1}$$

where x_D is the design model state, u is the control, and v is a white, Gaussian noise vector with zero mean. The measurements for this system are assumed to be linear combinations of the states and are modeled by

$$z_D = M_D x_D + w \tag{2}$$

where z_D is the measurement vector, and w is a white, Gaussian noise vector of zero mean. The system is assumed to be both controllable and observable.

The task at hand is the design of a suitable controller for this sytem. As a measure of the quality of candidate designs, a quadratic performance index is assumed. This index is of the

Received Dec. 27, 1984; revision received July 14, 1986.

*Director, Software Development Department. Member AIAA.

†Professor, Mechanical, Aerospace and Nuclear Engineering Department. Associate Fellow AIAA.

form

$$J = \lim_{T\to\infty} \frac{1}{T} E \int_0^T [x_D^T Q x_D + u^T R u] \mathrm{d}t \tag{3}$$

where the matrices Q and R are specified in the problem description. We will further assume that R is positive definite and Q is positive semidefinite.

The optimal solution to this problem (assuming no modeling error) is well known.[10,11] It consists of a control law and estimator which, in theory, can be designed independently. The control law

$$u = G_D^i \hat{x}_D \tag{4}$$

is based on linear feedback of the estimates of the design states. The gain

$$G_D = -R^{-1} B_D^T P_C \tag{5}$$

is obtained by solving a steady-state Riccati equation

$$A_D^T P_C + P_C A_D - P_C B_D R^{-1} B_D^T P_C + Q = 0 \tag{6}$$

The dynamics of the estimator are given by

$$\dot{\hat{x}}_D = A_D \hat{x}_D + B_D u + K_D (z_D - M_D \hat{x}_D) \tag{7}$$

with the estimator gain

$$K_D = P_E M_D^T R_E^{-1} \tag{8}$$

obtained from the filter form of the Riccati equation

$$A_D P_E + P_E A_D^T - P_E M_D^T R_E^{-1} M_D P_E + Q_E = 0 \tag{9}$$

This filter provides the best weighted least-squares estimate of x_D, given a known set of noise covariances. In terms of the design model, R_E is the covariance matrix of w, and Q_E is the covariance of $D_D v$, or equivalently,

$$Q_E = D_D Q_v D_D^T \tag{10}$$

where Q_v is the covariance matrix of v alone.

To account for the potential effects of spillover due to unmodeled, high-frequency modes, a modified optimization problem will be formulated. The spillover effects will be addressed by the specification of frequency domain constraints which appropriately attenuate the high frequency response of the full dynamic system.

Frequency Domain Constraints

Classical frequency domain techniques have traditionally been used to provide a decreased sensitivity in the closed-loop system, with respect to variations in the plant. Since the primary concern is the effects of uncertain high-frequency modes, a constraint based on a frequency domain characterization of plant uncertainty is well motivated.

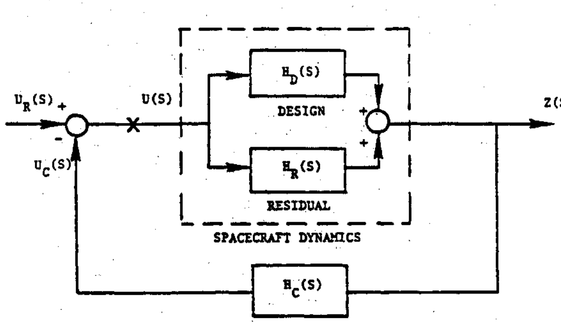

Fig. 1 Controlled spacecraft in frequency domain.

A paper by Kosut and Salzwedal[12] examined the selection of frequency domain measures of stability and robustness for the special problem of large space structures. The system under consideration is the one shown in Fig. 1, where the spacecraft dynamics have been broken into both design and residual modes. In this figure, the transfer functions for the design model, residual modes, and control system (including both control law and estimator) are given by $H_D(s)$, $H_R(s)$, and $H_C(s)$, respectively. The Laplace domain representations of the reference input, control input, and system output (measurements) are given by $U_R(s)$, $U_C(s)$ and $Z(s)$.

The specific form of the transfer functions can be determined from the problem statement. The design and residual modes are assumed to be dynamically uncoupled

$$\dot{x}_D = A_D x_D + B_D u \tag{11}$$

$$\dot{x}_R = A_R x_R + B_R u \tag{12}$$

$$z = M_D x_D + M_R x_R \tag{13}$$

and yield transfer functions given by

$$H_D(s) = M_D (sI_D - A_D)^{-1} B_D \tag{14}$$

$$H_R(s) = M_R (sI_R - A_R)^{-1} B_R \tag{15}$$

where the matrices I_D and I_R are appropriately dimensioned identity matrices and s is the Laplace variable.

For the transfer function of the controller, a linear feedback form is assumed

$$u = G\hat{x}_D \tag{16}$$

based upon an estimator for the design states

$$\dot{\hat{x}}_D = A_D \hat{x}_D + B_D u + K(z - M_D \hat{x}_D) \tag{17}$$

Note that this form is consistent with the optimal control law and estimator shown earlier, but the selection of the gains G and K is not limited to the optimal gains, G_D and K_D. With these equations, the closed-loop system transfer function becomes

$$H_C(s) = G(sI_D - A_D - B_D G + K M_D)^{-1} K \tag{18}$$

This frequency response description can provide guidelines for selecting G and K so as to decrease the sensitivity of the design to the residual modes. To do this, the feedback loop can be broken at the point indicated by the X in Fig. 1. If no residual modes were present, the open-loop transfer function at this point would simply be $H_C(s)H_D(s)$. The stability of the nominal system could then be guaranteed by requiring the eigenvalues of $A_D + B_D G$ and $A_D - KM_D$ to have negative real parts (i.e., a stable controller and estimator).

When residual modes are present, the stability of the closed-loop system is no longer guaranteed. Consideration of the perturbed system shows that its open-loop transfer function at the same point is $H_C(s)[H_D(s) + H_R(s)]$. Expressed as $H_C(s)H_D(s) + H_C(s)H_R(s)$, the second term can be identified as an additive perturbation to the nominal transfer function, $H_C(s)H_D(s)$. When model errors can be characterized in this fashion, frequency domain bounds can be developed for guaranteeing stability. For this work, the two robustness tests given by Kosut and Salzwedal[12] will be used, and are restated:

Robustness Test 1

$$\underline{\sigma}[I + H_C(j\omega)H_D(j\omega)] > \bar{\sigma}[H_C(j\omega)H_R(j\omega)] \tag{19}$$

for all $\omega > 0$, ensures stability provided $H_C(j\omega)H_D(j\omega)$ and $H_C(j\omega)H_R(j\omega)$ are stable. The terms $\bar{\sigma}[\cdot]$ and $\underline{\sigma}[\cdot]$ refer to the maximum and minimum singular values of a matrix.[13] These are defined in terms of the maximum and minimum eigenvalues of H^*H as

$$\bar{\sigma}[H] \triangleq [\lambda_{\max}(H^*H)]^{1/2} \tag{20}$$

$$\underline{\sigma}[H] \triangleq [\lambda_{\min}(H^*H)]^{1/2} \tag{21}$$

where (*) denotes complex conjugate transpose.

Robustness Test 2

$$\underline{\sigma}[I+H_C(j\omega)H_D(j\omega)]/\bar{\sigma}[H_C(j\omega)] > \bar{\sigma}[H_R(j\omega)] \tag{22}$$

for all $\omega>0$, ensures stability under the same conditions as above, provided $H_C(j\omega)$ is stable, i.e., all the eigenvalues of $A_D+B_DG-KM_D$ have negative real parts.

In interpreting these tests, it is instructive to consider a single-input single-output system. The quantities $H_C(j\omega)H_D(j\omega)$ and $H_C(j\omega)H_R(j\omega)$ are then scalars, and the singular values are just the absolute values of the scalars. Robustness Test 1 becomes

$$|1+H_C(j\omega)H_D(j\omega)| > |H_C(j\omega)H_R(j\omega)| \tag{23}$$

Now examine the classical test for system stability. The full-order closed-loop system will be stable if the roots of the characteristic equation have negative real parts. Equivalently, the poles of the closed-loop transfer function (i.e., the zeros of the denominator) must lie in the left half of the complex plane. Remembering that $H_CH_D+H_CH_R$ is the open-loop transfer function, the stability of the system requires

$$|1+H_C(j\omega)H_D(j\omega)+H_C(j\omega)H_R(j\omega)|>0 \tag{24}$$

for all $\omega>0$. Applying the triangle inequality

$$|a_1+a_2| \geq |a_1|-|a_2| \tag{25}$$

to the respective quantities yields

$$|1+H_C(j\omega)H_D(j\omega)+H_C(j\omega)H_R(j\omega)|$$
$$\geq |1+H_C(j\omega)H_D(j\omega)|-|H_C(j\omega)H_R(j\omega)| \tag{26}$$

Thus, a sufficient condition for Eq. (24) to hold is that

$$|1+H_C(j\omega)H_D(j\omega)|-|H_C(j\omega)H_R(j\omega)|>0 \tag{27}$$

for $\omega>0$. This is precisely the robustness test given in Eq. (23).

The multi-input results follow from generalizations of both the stability criterion and triangle inequality to the multi-input, multi-output case. The absolute values are replaced by matrix norms, as expressed by the singular values. The maximum and minimum singular values of a matrix are measures of its maximum and minimum "size," respectively. The robustness test given in Eq. (19) is thus comparing the minimum size of the nominal transfer function to the maximum size of the perturbation, at each frequency.

The second robustness test is more conservative and is implied by the first. However, the second test is also more useful in the space structure design synthesis problem. The right-hand side contains the residual modes transfer function which is, in general, unknown. On the other hand, the characteristics of the control system are known once the control gain, G, and filter gain, K, are chosen. For a specific set of gains and a given design model, the left side of (22) can be evaluated. By seeking gains so that this quantity is maximized (or at least greater than a specified value), a stable, robust design can be found. This idea forms the underlying concept behind many model error-compensation techniques (e.g., frequency-shaped cost functions,[14,15] filter-accommodated control[16]), as well as the basis for the current approach.

Problem Formulation and Solution

The problem to be solved can now be stated in terms of the optimization criteria and constraints. It is desired to find the gains G and K to be used in a control law

$$u=G\hat{x}_D \tag{28}$$

and estimator

$$\dot{\hat{x}}=A_D\hat{x}+B_Du+K(z-M_D\hat{x}_D) \tag{29}$$

so that a given cost functional

$$J=\lim_{T\to\infty}\frac{1}{T}E\int_0^T[x_D^TQx_D+u^TRu]\,dt \tag{30}$$

is minimized for the system

$$\dot{x}_D=A_Dx_D+B_Du+D_Dv \tag{31}$$

$$z_D=M_Dx_D+w \tag{32}$$

Furthermore, the solution is constrained to satisfy a frequency domain inequality of the form

$$\frac{\underline{\sigma}[I+G(j\omega I_D-A_D-B_DG+KM_D)^{-1}KM_D(j\omega I_D-A_D)^{-1}B_D]}{\bar{\sigma}[G(j\omega I_D-A_D-B_DG+KM_D)^{-1}K]}$$
$$>\gamma(\omega) \tag{33}$$

The function $\gamma(\omega)$ is chosen so that $\gamma(\omega)\geq\sigma[H_R(j\omega)]$ at all $\omega>0$.

The general form of the problem assumes that both the regulator and estimator gains will be determined in the solution. However, this formulation can also be used to determine the best control law for a specified estimator, or the best estimator for a specified control law. These approaches are appropriate, since it is the combination of control and observation spillover that causes loss of performance, and reducing either effect separately reduces their joint effect.[17] The two situations can be handled as special cases of the general technique.

One way of solving this problem is by parameter optimization. A set of problem unknowns are identified for which the optimization criteria and constraints can be evaluated. After specifying an initial guess for these unknowns, a digital computer algorithm is used to refine the guess until the optimal solution which satisfies the constraints is found.

For the problem identified in Eqs. (28-33), the most straightforward selection of unknowns is the gains, G and K. If the dimensions of the design model, inputs, and measurements are n, p, and m, respectively, the total number of unknowns will be np (for G) plus nm (for K). The number of unknowns increases only linearly with increases in the design model dimensions, and is small for low-order problems. Alternative formulations of the problem are given in Ref. 18.

In order to perform the optimization, the performance index given in Eq. (30) must be evaluated for a given set of gains. To do this, define

$$\bar{Q}=\begin{bmatrix} Q & 0 \\ 0 & G^TRG \end{bmatrix} \tag{34}$$

$$\bar{A}=\begin{bmatrix} A_D & B_DG \\ KM_D & A_D+B_DG-KM_D \end{bmatrix} \tag{35}$$

$$\bar{V}=\begin{bmatrix} D_D & 0 \\ 0 & K \end{bmatrix}\begin{bmatrix} Q_v & 0 \\ 0 & Q_w \end{bmatrix}\begin{bmatrix} D_D^T & 0 \\ 0 & K^T \end{bmatrix} \tag{36}$$

where Q_v and Q_w are the covariances of v and w, respectively. Then the performance index can be evaluated as[10]

$$J=\mathrm{tr}[\bar{P}\bar{V}] \tag{37}$$

where tr [·] denotes the trace of a matrix. The matrix $\bar{P}$ satisfies a Lyapunov equation

$$\bar{A}^T\bar{P}+\bar{P}\bar{A}+\bar{Q}=0 \tag{38}$$

which can be solved with existing computer routines.[10,19] Note that optimization of the expression given in Eq. (37) for the *unconstrained* case would yield the standard optimal LQG regulator and estimator.

Some difficulties may be encountered in applying this approach to multi-input multi-output examples. The singular value norms are nondifferentiable with respect to the parameters at points where the singular values are not distinct. This can cause problems in the parameter optimization because of the difficulty in computing gradients. Approaches have been proposed for dealing with this problem.[20,21] Alternatively, it is possible to formulate the inqualities with matrix norms other than the singular values.[22] For this paper, the numerical example will be restricted to single-input single-output. Applications with multi-input multi-output systems will be addressed in future papers.

Numerical Example

In order to demonstrate the design synthesis techniques outlined, a numerical example will be presented. The intent of this example is to illustrate some of the concepts and tradeoffs inherent in the large, space structure controller design problem. However, in order to avoid the complexities involved in the myriad configurations presently being studied, a more generic structure will be used.

The system under study is the simple, uniform beam illustrated in Fig. 2. Both ends of the beam are mounted in guides which permit translational motion relative to the support, but maintain zero slope at both ends. The dynamics under consideration are those of the beam's transverse displacement at points along the beam as measured from a horizontal reference line. This displacement is denoted as $y(x,t)$, where x is the distance along the beam measured from the left end, and t is time. The beam parameters M (mass per unit length), I (moment of inertia), and E (modulus of elasticity) are assumed to be constant throughout. The length of the beam is L.

The beam dynamics are modeled by the Euler-Bernoulli partial differential equation

$$My_{tt}(x,t)+EIy_{xxxx}(x,t)=F(x,t) \tag{39}$$

where the subscripts t and x indicate differentiation with respect to the indicated variable. The quantity $F(x,t)$ represents the force exerted on the beam, as a function of position and time.

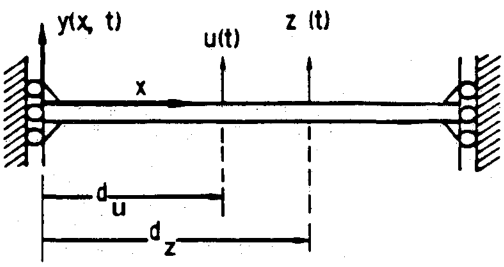

Fig. 2 Simple beam with guided end conditions.

The mass properties of the beam are chosen, primarily for convenience, as $M=EI=2/\pi$ and $L=\pi$. With these values, the deflection is given by

$$y(x,t)=(\sqrt{2}/2)q_0(t)+\sum_{n=1}^{\infty}\cos(nx)q_n(t) \tag{40}$$

where $q_n(t)$ satisfies

$$\ddot{q}_0(t)=(\sqrt{2}/2)(\pi/2)u(t) \tag{41}$$

$$\ddot{q}_n(t)+2\xi\omega_n\dot{q}_n(t)+\omega_n^2q_n(t)=\cos(nd_u)(\pi/2)u(t), \quad n=1,2,\ldots \tag{42}$$

A time-varying control force, $u(t)$ is assumed to be acting at a distance, d_u, as indicated in Fig. 2. The modal frequencies are given by $\omega_n=n^2$, and an assumed model damping value of $\xi=0.005$ will be used.

The objective of the control problem will be to keep the beam close to the reference position, while minimizing the energy in the modes and the control energy expended. For simplicity, a single force actuator and single position sensor will be used. To maintain controllability and observability, and to adequately illustrate the effects of spillover, it is necessary to locate the actuator and sensor sufficiently far from the nodes of the flexible modes. In view of the modeshapes, an appropriate position for both is at the left end of the beam (i.e., $d_u=0$). Every mode is both controllable and observable at this location.

The equations of motion for the design model can now be written. Consistent with the previous discussions on modeling, only the lowest-frequency modes will be retained in this model. For this example, this will consist of the rigid body mode and first two flexible modes, yielding a sixth-order model in the form

$$\dot{x}_D=A_Dx_D+B_Du+D_Dv \tag{43}$$

$$z_D=M_Dx_D+w \tag{44}$$

Table 1 LQG controller eigenvalues

	Eigenvalues	Frequency	Damping
Regulator	$-1.9668 \pm i\,1.8360$	2.6869	0.7320
	$-0.9436 \pm i\,0.9822$	0.9867	0.0956
	$-0.6262 \pm i\,3.9626$	4.0118	0.1561
Estimator	$-0.1337 \pm i\,3.9626$	0.5630	0.2375
	$-0.6193 \pm i\,1.0957$	1.2586	0.4920
	$-0.1265 \pm i\,3.9896$	3.9916	0.0317

Table 2 Evaluation model eigenvalues, LQG controller

	Eigenvalues	Frequency	Damping
Design modes	$-0.1312 \pm i\,0.3422$	0.5579	0.2351
	$-0.0943 \pm i\,0.9816$	0.9861	0.0956
	$-0.5845 \pm i\,1.1548$	1.2943	0.0452
	$-2.2170 \pm i\,1.2192$	2.6324	0.8422
	$-0.1293 \pm i\,3.9923$	3.9944	0.0324
	$-0.5479 \pm i\,3.9416$	3.9795	0.1377
Residual modes	$0.0311 \pm i\,9.0781$	9.0781	-0.0034
	$-0.0473 \pm i\,16.0143$	16.0144	0.0030
	$-0.1108 \pm i\,25.0035$	25.0037	0.0044
	$-0.1730 \pm i\,36.0008$	36.0012	0.0048
	$-0.2412 \pm i\,49.0000$	49.0005	0.0049
	$-0.3178 \pm i\,63.9994$	64.0002	0.0050
	$-0.4036 \pm i\,80.9991$	81.0001	0.0050

where

$$x_D = [q_0 \ \dot{q}_0 \ \dot{q}_1 \ q_1 \ q_2 \ \dot{q}_2]^T \tag{45}$$

$$A_D = \begin{bmatrix} 0.000 & 1.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 1.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & -1.000 & -0.010 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 1.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & -16.000 & -0.040 \end{bmatrix} \tag{46}$$

$$B_D = [0.000 \quad 0.717 \quad 0.000 \quad 1.000 \quad 0.000 \quad 1.000]^T \tag{47}$$

$$M_D = [0.717 \quad 0.000 \quad 1.000 \quad 0.000 \quad 1.000 \quad 0.000] \tag{48}$$

The noise in the state equation is assumed to come from noise in the actuator (i.e., $D_D = B_D$).

The performance index chosen for controller design must reflect the objectives of the control problem, i.e., keeping the beam close to the reference position, minimizing energy in the modes, and minimizing control energy. To penalize the deviation of the beam from the reference, it is sufficient to penalize the square of the rigid body position, q_0. The energy in the modes is a combination of kinetic energy and potential energy:

$$\text{Energy} = \sum_{n=0}^{2} \dot{q}_n^2 + \sum_{n=0}^{2} \omega_n^2 q_n^2 \tag{49}$$

Minimization of the control energy is handled by a penalty on the square of the control effort.

The three components can be combined within the framework of the standard LQG infinite-time regulator problem

$$J = \lim_{T\to\infty} \frac{1}{T} E \int_0^T [x_D^T Q x_D + u^T R u] \, dt \tag{50}$$

with $R = 1$ and $Q = \text{diag}[100., 1., 1., 1., 16., 1.]$. Note that the beam rigid body position is weighted by a factor of 100. This particular selection of weightings was chosen so as to adequately illustrate the effects of spillover.

The standard LQG controller for this system takes the form of a feedback control law based on estimates supplied by a Kalman filter. The noise sources are assumed to be uncorrelated with unitary covariances, i.e., $E\{vv^T\} = 1$, $E\{ww^T\} = 1$. The control gain G_D and filter gain K_D are then given numerically by

$$G_D = [-10.000 \quad -8.040 \quad -0.942 \quad -0.947 \quad -4.927 \quad -0.586] \tag{51}$$

$$K_D = [\ 1.152 \quad 0.707 \quad 0.692 \quad 0.702 \quad 0.203 \quad 0.258]^T \tag{52}$$

Using these gains, the value of the performance index given in Eq. (50) is determined to be 366.18. The eigenvalues of the resulting closed-loop system are given in Table 1.

These regulator and estimator designs are optimal for the sixth-order design model. Of greater interest however, is the performance of this controller in the full-order system, including the residual modes. To make this assessment, a finite set of residual modes must be identified. For this example, the residual system will be chosen as the next seven lowest frequency modes. The evaluation model will therefore consist of the first ten modes of the beam.

If no control or observation spillover were present, the eigenvalues of the full-order system would be the eigenvalues of the regulator, estimator, and the open-loop residual modes. However, these eigenvalue locations are perturbed by the effects of spillover. The eigenvalues resulting from implementing the LQG gains in the full-order system are given in Table 2.

Note that the full-order system is actually *destabilized* by the optimal LQG controller. In particular, the first residual mode eigenvalue has been shifted into the right half-plane. This is because the control action applied at the left end of the beam excites this mode (control spillover), and the corresponding motions are seen in the position measurements (observation spillover). From a stability standpoint, this controller design is clearly unacceptable.

Robust Controller Design Example

In the previous section, it was shown that the standard LQG technique for designing reduced-order controllers may produce controllers that destabilize the full-order system. In this section, the concept of frequency-shaped controller design is illustrated.

The frequency domain quantities of Eq. (22) are illustrated in Figs. 3 and 4 for the previous LQG design. One can see that the frequency domain robustness condition does not hold, with the violation occurring at the peak of the first residual mode, near 9 rad/s. To address this, a constraint of the form in Eq. (33) will be written. In a typical design problem, some

Table 3 Evaluation model eigenvalues, frequency-shaped regulator and estimator

	Eigenvalues	Frequency	Damping
Design modes	$-0.1368 \pm i\, 0.5381$	0.5545	0.2411
	$-0.0956 \pm i\, 0.9698$	0.9745	0.0981
	$-0.6523 \pm i\, 0.9416$	0.1455	0.5695
	$-0.6835 \pm i\, 1.6555$	1.7910	0.3816
	$-0.1893 \pm i\, 3.8755$	3.8801	0.0488
	$-0.1171 \pm i\, 4.0338$	4.0355	0.0290
Residual modes	$0.0089 \pm i\, 9.0148$	9.0148	0.0011
	$-0.0680 \pm i\, 16.0023$	16.0024	0.0042
	$-0.1201 \pm i\, 25.0003$	25.0006	0.0048
	$-0.1776 \pm i\, 35.9998$	36.0002	0.0049
	$-0.2437 \pm i\, 48.9995$	49.0001	0.0050
	$-0.3192 \pm i\, 63.9992$	64.0000	0.0050
	$-0.4045 \pm i\, 80.9990$	81.0000	0.0050

sort of approximation would be used for the bounding function $\gamma(\omega)$ for the residual modes (or other modeling errors). For simplicity, this example will use the constraint

$$[1+H_C(j\omega)H_D(j\omega)|/|H_C(j\omega)||>1.4125 \tag{53}$$

applied at $\omega=9$ rad/s. The value of 1.4125 (3dB) is chosen so as to sufficiently bound the first residual mode. The bounding function would generally be a function of frequency, but applying this constraint at this one frequency was found to be sufficient for this problem. The weighting matrices of the performance index are the same as those specified for the LQG controller.

This problem was solved via a standard, constrained parameter optimization solution package.[23,24] The optimal set of gains satisfying the frequency domain constraint were found to be

$$G=[-4.440 \quad -2.600 \quad -0.399 \quad 0.359 \quad 0.138 \quad -0.347] \tag{54}$$

$$K=[\ 1.021 \quad 0.532 \quad 0.812 \quad 0.442 \quad 0.167 \quad 0.560]^T \tag{55}$$

The value of the performance index for these gains is $J=401.32$. This is only a modest increase compared to the LQG design ($J=366.18$). However, Fig. 5 shows that the frequency domain constraint is now met by this design. Hence, asymptotic stability of the full-order system is guaranteed, as can be verified from the eigenvalues of the evaluation model (Table 3).

Note that this technique offers significant latitude in selection of the controller gains. The only requirements placed on the gains is that they stabilize the controller transfer function

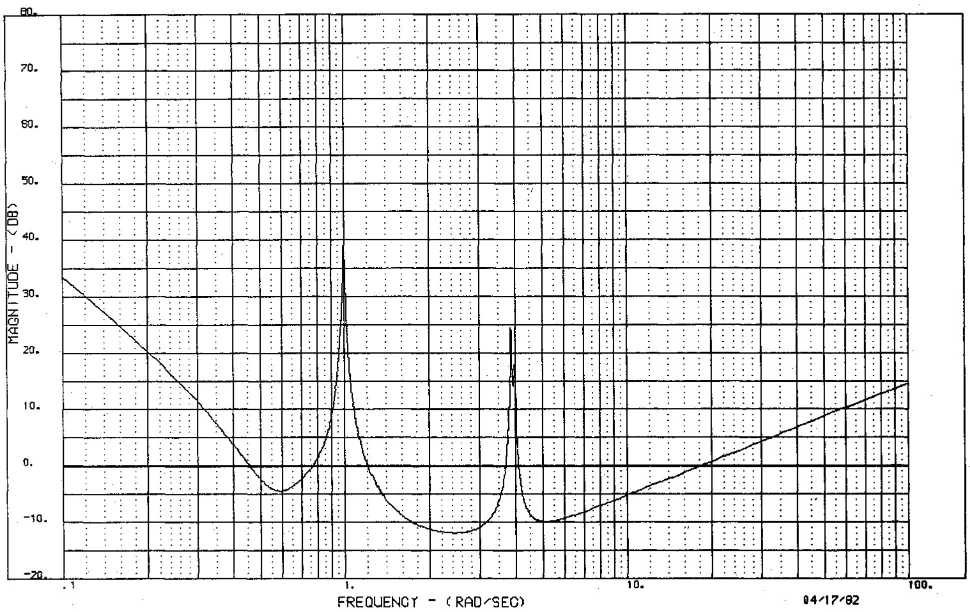

Fig. 3 $\underline{\sigma}|1+H_C(j\omega)|H_D(j\omega)|/|H_C(j\omega)|$ **for LQG controller.**

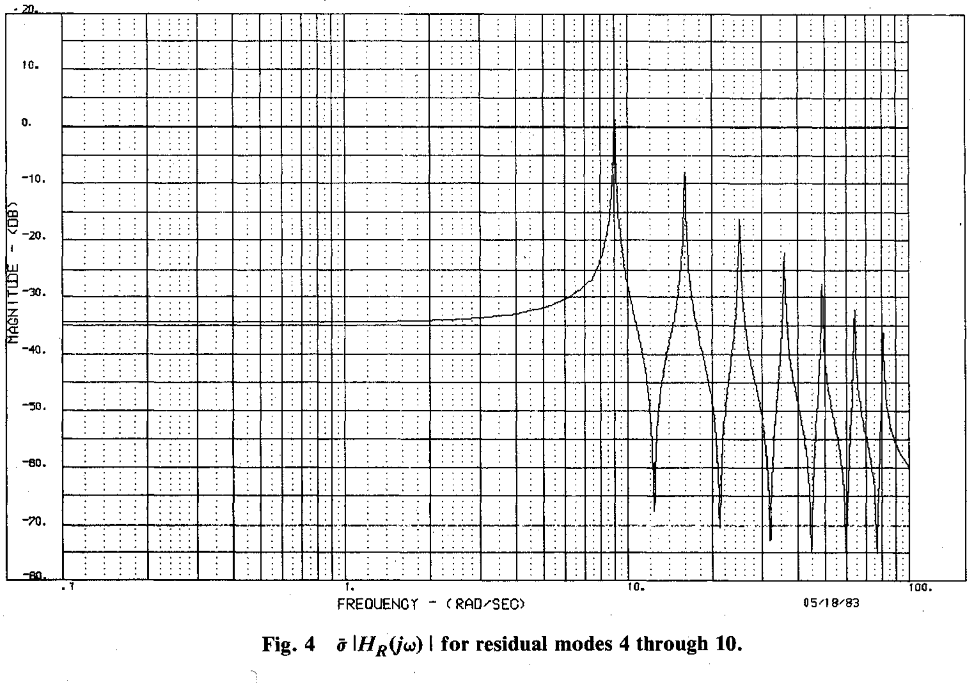

Fig. 4 $\bar{\sigma}|H_R(j\omega)|$ **for residual modes 4 through 10.**

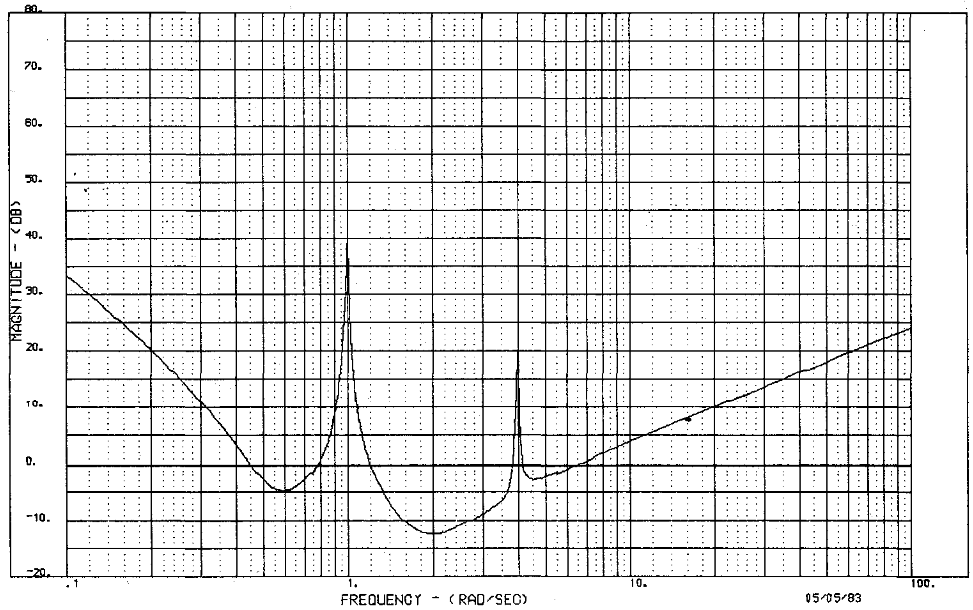

Fig. 5 $|1+H_C(j\omega)H_D(j\omega)|/|H_C(j\omega)|$ **for frequency-shaped regulator and estimator.**

and satisfy the frequency domain criteria. This permits significant optimization of the performance index over a wide set of gains. In fact, it can be shown[18,25] that the gains computed for this example are not even in the class of optimal LQG gains, i.e., they do not satisfy a Riccati equation for any selection of weighting matrices. Nevertheless, they are the optimal solution for the constrained problem at hand.

To further illustrate this point, a much simpler design approach was tried. Using the LQG formulation, the control penalty was increased until a design was found that met the frequency domain constraint. The resulting design had a performance index of $J=440.44$ which is significantly poorer than the more general solution ($J=401.32$). This illustrates that the increased robustness comes from changes in the control problem emphasis, not just a general reduction in control bandwidth.

Concluding Remarks

A procedure was presented in this paper for the design of robust controllers in the presence of modeling errors. The approach combines a quadratic cost functional to measure performance with frequency domain constraints for robustness. The frequency domain constraints are chosen so as to guarantee stability of the full-order closed-loop system for a bounded class of residual modes. The resulting problem is formulated and solved as a nonlinear, constrained, parameter optimization problem.

The technique was applied to the design of a controller for a flexible beam. A standard LQG controller was found to yield an unstable full-order closed-loop system due to the effects of control and observation spillover. When the frequency domain constraint was applied, a stable controller was found with only a modest increase in the value of the cost functional.

References

[1]Skelton, R.E. and Likins, P.W., "Techniques of Modeling and Model Error Compensation in Linear Regulator Problems," *Control and Dynamic Systems,* edited by C.T. Leondes, Vol. 14, Academic Press, New York, 1978.

[2]Zienkiewcs, O., *The FEM in Engineering Science,* McGraw-Hill, New York, 1971.

[3]Strang, G. and Fix., G., *An Analysis of the FEM,* Prentice-Hall, Englewood Cliffs, NJ, 1976.

[4]Oden, J.T. and Reddy, J., *An Introduction to the Mathematical Theory of FE,* Wiley, New York, 1976.

[5]Richtmyer, R. and Morton K., *Difference Methods for Initial Value Problems,* Second Ed., Wiley, New York, 1967.

[6]Gottlieb, D. and Orszag, S., *Numerical Analysis of Spectral Methods: Theory and Applications,* SIAM Regional Conference Series in Applied Math 26, 1977.

[7]Meirovitch, L., *Computational Methods in Structural Dynamics,* Sijthoff-Noordhoff International Publishers, Alphen nan den Rijn, the Netherlands, 1980.

[8]Balas, M.J., "Active Control of Flexible Systems," *Journal of Optimization Theory and Applications,* Vol. 25, No. 3, July 1978, pp. 415-436.

[9]Balas, M.J., "Feedback Control of Flexible Systems," *IEEE Transactions on Automatic Control,* Vol. AC-23, No. 4, 1978, pp. 673-679.

[10]Kwakernak, H. and Sivan, R., *Linear Optimal Control Systems,* Wiley-Interscience, New York, 1972.

[11]Anderson, B.D.O. and Moore, J.B., *Linear Optimal Control,* Prentice-Hall, Englewood Cliffs, NJ, 1971.

[12]Kosut, R.L. and Salzwedal, H., "Stability and Robustness of Control Systems for Large Space Structures," *Proceedings of the Third VPI&SU/AIAA Symposium on Dynamics and Control of Large Flexible Spacecraft,* Blacksburg, VA, June 15-17, 1981, pp. 343-364.

[13]Stewart, G.W., *Introduction to Matrix Computations,* Academic Press, New York, 1973.

[14]Gupta, N.K., "Frequency-Shaped Cost Functionals: Extension of Linear-Quadratic-Gaussian Design Methods," *Journal of Guidance and Control,* Vol. 3, Nov.-Dec. 1980, pp. 529-535.

[15]Gupta, N.K., "Robust Control/Estimator Design by Frequency-Shaped Cost Functionals," *Proceedings of the 20th IEEE Conference on Decision and Control Including the Symposium on Adaptive Processes,* Paper No. 1167-72, Vol. 3, San Diego, CA, Dec. 16-18, 1981.

[16]Sesak, J.R., Halstenberg, R.V., Chang, Y., and Davis, M.M., "Filter-Accommodated Optimal Control of Large Flexible Space Systems," AIAA Paper 81-1784, Aug. 1981.

[17]Skelton, R.E., "Control Design of Flexible Spacecraft," *Theory and Applicatons of Optimal Control in Aerospace Systems,* (Chap. 8), edited by I.P. Kant. Agardograph #251, Neuilly sur Seine, France, Advisory Group for Aerospace Research & Development, July 1981.

[18]Hefner, R.D., "Robust Controller Design using Frequency Domain Constraints: Alternative Problem Formulations," *Proceedings of the Fourth VPI&SU Symposium on Dynamics and Control of Large Structures,* Blacksburg, VA, June 6-8, 1983, pp. 549-566.

[19]Bartels, R.H. and Stewart, G.W., "Solution of the Matrix Equation $AX+XB=C$," *Communications of the ACM,* Vol. 15, No. 9, Sept. 1972, pp. 820-826.

[20]Polak, E. and Mayne, D.Q., "On the Solution of Singular Value Inequalities Over a Continuum of Frequencies," *Proceedings of the*

19th IEEE Conference on Decision and Control, Vol. 1, Dec. 1980, pp. 23–28.

[21]Polak, E. and Wardi, Y., "Nondifferentiable Optimization Algorithm for Designing Control Systems Having Singular Value Inequalities," *Automatica,* Vol. 18, No. 3, May 1982, pp. 267–283.

[22]Laub, A.J., "Efficient Multivariable Frequency Response Computations," *Proceedings of the 19th IEEE Conference on Decision and Control,* Vol. 1, Dec. 1980, pp. 39–41.

[23]"The Generalized Trajectory Simulation System, Volume IV: Numerical Operators," TR-0076 (666)-1, The Aerospace Corporation, El Segundo, CA, 1976.

[24]Betts, J.T., "An Accelerated Multiplier Method for Nonlinear Programming," TR-007 (5901-02)-5, The Aerospace Corporation, El Segundo, CA, 1974.

[25]Hefner, R.D., "Suboptimal Controller Design Using Frequency Domain Constraints," Ph.D. dissertation, University of California, Los Angeles, 1982.